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## Primary scalar hair in dilatonic theories with modulus fields

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### ABSTRACT

We study the general spherical symmetric solutions of dilaton-modulus gravity non-minimally coupled to a Maxwell field, using methods from the theory of dynamical systems. We show that the solutions can be classified by the mass, the electric charge, and a third parameter which we argue can be related to a scalar charge. The global properties of the solutions are discussed.

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## Introduction.

It is known that black hole solutions of charged dilaton-gravity models, as those arising in effective string theory, present a quite different behaviour from the Reissner-Nordström solutions of general relativity [1-2]. This is essentially due to the non-minimal coupling of the dilaton to the Maxwell field, which spoils the validity of the standard no-hair theorems [3] and hence allows for the presence of a non-trivial dilaton field. In spite of this, the dilaton charge is not an independent parameter, but is still a function of the mass and the electric charge of the black hole and has henceforth sometimes been called a secondary hair.

In effective four-dimensional string theory, however, further scalar fields are present besides the dilaton, as for example the moduli coming from compactification of higher dimensions, which are non-minimally coupled to the Maxwell field [4]. The introduction of these fields may change the properties of the black holes. A simplified model which takes into account one modulus has been studied some time ago [5]. It was shown that an exact spherically symmetric black hole solution of the field equations can be found by requiring that the dilaton and the modulus are proportional. However, this restriction is not necessary, and it would be interesting to investigate the properties of the most general spherically symmetric solutions. In general, it is not possible to find these solutions in analytic form. (The field equations can in fact be cast in the form of a Toda molecule system of first order differential equations, which is exactly solvable only in a few special cases). However, the qualitative behaviour of the solutions and some quantitative results can be obtained by studying the Toda dynamical system. In particular, the metric and the scalar fields will necessarily be regular at all the points of the integral curves except critical points. Consequently, in order to determine the global properties of the solutions, as the structure of their horizons and asymptotic regions, it suffices to study their behaviour at the critical points of the dynamical system. One drawback of this method is that only the exterior region of the black hole can be studied. The interior may be however investigated numerically by continuing the solutions beside the horizon.

In this paper we undertake the investigation of the general solutions of the model introduced in [5] using this approach, and show that in general there exists a three-parameter family of asymptotically flat black hole solutions. This result is interesting because the third parameter can be presumably related to a scalar charge, giving therefore an example of primary scalar hair. In addition to these solutions, the model also admits as a limiting case a two-parameter family of non-asymptotically flat black hole degenerate solutions of the kind discussed in [6]. We also discuss the properties of extremal black hole solutions, which are of great interest in recent developments of string and membrane theories.

The paper is organized as follows. In section 1 we describe the model and obtain the dynamical system associated with the field equations. In section 2 we discuss the exact black hole solutions, obtained for special values of the parameters. In section 3 we study the dynamical system in its generality, while in section 4 we discuss the physical properties of its solutions.

## 1. The action and the field equations.

We study the action [5]:

$$S = \int d^4x \sqrt{-g} \left[ R - 2(\nabla\Phi)^2 - \frac{2}{3}(\nabla\Sigma)^2 - (e^{-2\Phi} + \lambda^2 e^{-2q\Sigma/3})F^2 \right] \quad (1.1)$$

where  $\Phi$  and  $\Sigma$  are the 4-dimensional dilaton and modulus respectively,  $F$  is the Maxwell field strength, and  $q$  and  $\lambda$  are coupling parameters. This action has been obtained by dimensional reduction of heterotic string effective action [7], with the addition of a non-minimal coupling term for the modulus, arising from integrating out heavy modes [4].

The field equations ensuing from (1.1) are

$$\begin{aligned} R_{\mu\nu} &= 2\nabla_\mu\Phi\nabla_\nu\Phi + \frac{2}{3}\nabla_\mu\Sigma\nabla_\nu\Sigma + 2(e^{-2\Phi} + \lambda^2 e^{-2q\Sigma/3}) \left( F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}F^2 g_{\mu\nu} \right) \\ \nabla^\mu[(e^{-2\Phi} + \lambda^2 e^{-2q\Sigma/3})F_{\mu\nu}] &= 0 \\ \nabla^2\Phi &= -\frac{1}{2}e^{-2\Phi}F^2 \\ \nabla^2\Sigma &= -\frac{q\lambda^2}{2}e^{-2q\Sigma/3}F^2 \end{aligned} \quad (1.2)$$

A spherically symmetric solution can be found with Maxwell field strength

$$F_{mn} = Q\epsilon_{mn} \quad m, n = 2, 3 \quad (1.3a)$$

and a metric of the form [1]

$$ds^2 = e^{2\nu}(-dt^2 + e^{4\rho}d\xi^2) + e^{2\rho}d\Omega^2 \quad (1.3b)$$

where  $\nu$ ,  $\rho$ ,  $\Phi$  and  $\Sigma$  are functions of the "radial" coordinate  $\xi$ . Defining a new function  $\zeta = \nu + \rho$ , the field equations (2) take the simpler form

$$\zeta'' = e^{2\zeta} \quad (1.4a)$$

$$\Phi'' = -Q_1^2 e^{2\nu-2\Phi} \quad (1.4b)$$

$$\Sigma'' = -qQ_2^2 e^{2\nu-2q\Sigma/3} \quad (1.4c)$$

$$\nu'' = Q_1^2 e^{2\nu-2\Phi} + Q_2^2 e^{2\nu-2q\Sigma/3} \quad (1.4d)$$

with  $Q_1^2 = Q^2$  and  $Q_2^2 = \lambda^2 Q^2$ , subject to the constraint

$$\zeta'^2 - \nu'^2 - \Phi'^2 - \frac{1}{3}\Sigma'^2 + Q_1^2 e^{2\nu-2\Phi} + Q_2^2 e^{2\nu-2q\Sigma/3} - e^{2\zeta} = 0 \quad (1.5)$$

A first integral of eq.(1.4a) is given by

$$\zeta'^2 = e^{2\zeta} + a^2$$

where  $a^2$  is an integration constant, which has been chosen to be non-negative because otherwise one would obtain solutions with no asymptotic region, which are not of interest to us. For the moment we consider only strictly positive values of  $a$ . As we shall see, the limit  $a \rightarrow 0$ , corresponds to extremal solutions. Integrating again, with a suitable choice of the origin of  $\xi$ , one gets

$$e^\zeta = \frac{2ae^{a\xi}}{1 - e^{2a\xi}} \quad (1.6)$$

where  $a$  can be chosen to be positive without loss of generality. Moreover, from the remaining eqs. (1.4), one obtains the relation

$$\frac{1}{q}\Sigma'' + \Phi'' + \nu'' = 0$$

which can be integrated, to read

$$\Sigma' = -q(\nu' + \Phi' + c) \quad (1.7)$$

with  $c$  an integration constant. In view of (1.4) and (1.7), defining

$$\chi = \nu - \Phi \quad \eta = \nu - \frac{q}{3}\Sigma \quad (1.8)$$

the field equations can be put in the "Toda molecule" form

$$\begin{aligned} \chi'' &= 2Q_1^2 e^{2\chi} + Q_2^2 e^{2\eta} \\ \eta'' &= Q_1^2 e^{2\chi} + \frac{3+q^2}{3} Q_2^2 e^{2\eta} \end{aligned} \quad (1.9)$$

In terms of  $\chi$  and  $\eta$ , the derivatives of the fields  $\Phi$ ,  $\Sigma$  and  $\nu$  are given by

$$\begin{aligned} \Phi' &= \frac{3}{3+2q^2} \left( \eta' - \frac{3+q^2}{3} \chi' - \frac{q^2}{3} c \right) \\ \Sigma' &= \frac{3q}{3+2q^2} (\chi' - 2\eta' - c) \\ \nu' &= \frac{3}{3+2q^2} \left( \eta' + \frac{q^2}{3} \chi' - \frac{q^2}{3} c \right) \end{aligned} \quad (1.10)$$

and eq.(1.5) can be written

$$a^2 - \frac{3}{3+2q^2} \left[ \frac{3+q^2}{3} \chi'^2 + 2\eta'^2 - 2\eta' \chi' + \frac{q^2}{3} c^2 \right] + Q_1^2 e^{2\chi} + Q_2^2 e^{2\eta} = 0 \quad (1.11)$$

The equations (1.9) with the constraint (1.11) can be solved exactly in a few special cases, which are reported in the following section.

In the general case, they can be recast in the form of a 3-dimensional system of first-order differential equations. If we define the variables

$$X = \chi', \quad Y = \eta', \quad Z = |Q_2|e^\eta$$

then the constraint (1.10) can be considered as a definition of  $e^{2\chi}$ . Eliminating the term  $e^{2\chi}$  from eqs. (1.9), one obtains the system:

$$\begin{aligned} X' &= Z^2 + 2P(X, Y, Z) \\ Y' &= \frac{3+q^2}{3}Z^2 + P(X, Y, Z) \\ Z' &= YZ \end{aligned} \tag{1.12}$$

where

$$P(X, Y, Z) = Q_2^2 e^{2\chi} = \frac{1}{3+2q^2} [(3+q^2)X^2 + 6Y^2 - 6XY - 3B] - Z^2 \tag{1.13}$$

with  $B = \frac{3+2q^2}{3}a^2 - \frac{q^2}{3}c^2$ .

## 2. Exact solutions.

*A. The  $Q_2 = 0$  case.*

This limit case corresponds to minimal coupling of  $\Sigma$ , i.e.  $\lambda \rightarrow 0$ . By the no-hair theorem, the regular solutions should have constant  $\Sigma$ , as we shall verify. When  $Q_2 = 0$ , the equations (1.9) take the form

$$\chi'' = 2Q_1^2 e^{2\chi}, \quad \eta'' = Q_1^2 e^{2\chi} \tag{2.1}$$

The first equation can be integrated to give

$$\chi'^2 = 2Q_1^2 e^{2\chi} + b^2 \tag{2.2}$$

with  $b$  an integration constant. Moreover, comparing the two equations (2.1),

$$\eta' = \frac{1}{2}(\chi' - k) \tag{2.3}$$

with  $k$  an arbitrary constant. The constraint equation (1.11) becomes then

$$a^2 - \frac{b^2}{2} - \frac{3}{3+2q^2} \left[ \frac{1}{2}k^2 + \frac{q^2}{3}c^2 \right] = 0 \tag{2.4}$$

Integrating again (2.2), one gets

$$Q_1 e^\chi = \frac{\sqrt{2}bAe^{b\xi}}{1 - A^2 e^{2b\xi}} \tag{2.5}$$

with  $A$  an integration constant.

From these results and the relations (1.10), one can now write down the general solution in terms of the physical fields. Rather than giving all the explicit expressions, let us first consider the "radial" metric function  $e^\rho = e^{\zeta-\nu}$ . As  $\xi \rightarrow 0$ ,  $e^\rho \rightarrow \infty$ , and hence one can identify this limit with spatial infinity. As  $\xi \rightarrow -\infty$ , instead, one has from (1.6), (1.10) and (2.5),

$$e^\rho \sim \text{const} \times \exp \left[ \left( a - \frac{b}{2} + \frac{3}{3+2q^2} \left( \frac{1}{2}k + \frac{q^2}{3}c \right) \right) \xi \right] \quad (2.6)$$

which implies that for  $\xi \rightarrow -\infty$ ,  $e^\rho \rightarrow 0$ , giving rise to a singularity, except in the special case when the constant factor in the exponential vanishes, in which case  $e^\rho \rightarrow \text{const}$  as  $\xi \rightarrow -\infty$ . In conjunction with (2.4), this request singles out a unique real solution for the parameters, given by  $a = b = -c = -k$ .

In order to analyze the metric, it is useful to write it in a Schwarzschild-like form, by introducing a new radial coordinate  $r$ , such that  $dr = e^{2\zeta} d\xi$ . In the new coordinates ,

$$ds^2 = -e^{2\nu} dt^2 + e^{-2\nu} dr^2 + e^{2\rho} d\Omega^2 \quad (2.7)$$

where the metric functions are now viewed as functions of  $r$ . With a suitable choice of the origin of  $r$ , one has then

$$e^{2a\xi} = \frac{r - r_+}{r - r_-}, \quad e^{2\zeta} = (r - r_+)(r - r_-), \quad 1 - A^2 e^{2a\xi} = (1 - A^2) \frac{r}{r - r_-} \quad (2.8)$$

with  $r_+ = 2a/(1 - A^2)$ ,  $r_- = 2aA^2/(1 - A^2)$ . Moreover, if one chooses  $A$  such that  $Q_1 = 2aA/(1 - A^2)$ , the physical fields read, in terms of the new radial coordinate,

$$\begin{aligned} e^{2\nu} &= 1 - \frac{r_+}{r} & e^{2\rho} &= r^2 \left( 1 - \frac{r_-}{r} \right) \\ e^{-2\Phi} &= 1 - \frac{r_-}{r} & e^{-2\Sigma} &= \text{const} \end{aligned} \quad (2.9)$$

This is nothing but the well-known GHS solution [1-2]. It describes asymptotically flat black holes with mass  $r_+/2$  and charge  $Q_1^2 = r_+ r_-/2$ . The surface  $r = r_+$  is a horizon while the point  $r = r_-$  is a singularity.

Qualitatively different solutions arise in the special case  $A = 1$ . In this case,  $e^{2\zeta} \sim e^{2\chi}$ , and choosing the origin of  $r$  such that  $r_+ = 2a$ , one gets

$$\begin{aligned} e^{2\nu} &= r - r_+ & e^{2\rho} &= r \\ e^{-2\Phi} &= r & e^{-2\Sigma} &= \text{const} \end{aligned} \quad (2.10)$$

This solution is not asymptotically flat , but still possesses a horizon at  $r_+$  and is singular at the origin. It has been investigated in detail in ref. [6].

Another important limit is reached when  $a = 0$  and corresponds to extremal black holes with  $r_- = r_+$ . In fact, in that case  $e^{2\zeta} = \xi^{-2}$ , and proceeding as before one can

show that the existence of a regular horizon implies that also the parameters  $b$ ,  $c$  and  $k$  must vanish. Hence, one has  $e^{2\chi} = (\xi + A)^{-2}$ , with  $A$  an integration constant. Defining a new coordinate  $r = A(1 - A\xi^{-1})$ , the metric functions can be finally cast in the form (2.9), with  $r_+ = r_- = A$ .

*B. The  $Q_1 = 0$  case.*

This case corresponds to minimal coupling of  $\Phi$  and can be considered the limit of (1.1) for  $\lambda \rightarrow \infty$ . By the no-hair theorem, the regular solutions must have constant  $\Phi$ . The field equations are now

$$\chi'' = Q_2^2 e^{2\eta}, \quad \eta'' = \frac{3+q^2}{3} Q_2^2 e^{2\eta} \quad (2.11)$$

Proceeding as before, one gets

$$Q_2 e^\eta = \sqrt{\frac{3}{3+q^2}} \frac{2bAe^{b\xi}}{1 - A^2 e^{2b\xi}} \quad (2.12)$$

$$\chi' = \frac{3}{3+q^2} (\eta' - k)$$

where  $b$ ,  $A$ ,  $k$  are integration constants, together with the constraint

$$a^2 - \frac{3}{3+q^2} b^2 - \frac{3}{3+2q^2} \left[ \frac{3}{3+q^2} k^2 + \frac{q^2}{3} c^2 \right] = 0 \quad (2.13)$$

We look again for regular black hole solutions. For this purpose, we consider the asymptotic behaviour of  $e^\rho$  as  $\xi \rightarrow -\infty$ , which is now

$$e^\rho \sim \text{const} \times \exp \left[ \left( a - \frac{3}{3+q^2} b + \frac{3}{3+2q^2} \left( \frac{q^2}{3+q^2} k + \frac{q^2}{3} c \right) \right) \xi \right] \quad (2.14)$$

A horizon can only occur when the coefficient of  $\xi$  in the exponential vanishes, in which case  $e^\rho \rightarrow \text{const}$  as  $\xi \rightarrow -\infty$ . This condition, together with (2.13) implies that  $a = b = -c = -3k/q^2$ .

Defining a new coordinate  $r$  as before, for  $A$  such that  $Q_2 = \sqrt{\frac{3}{3+q^2}} \frac{2aA}{1-A^2}$ , one gets finally

$$e^{2\nu} = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right)^{\frac{3-q^2}{3+q^2}} \quad e^{2\rho} = r^2 \left( 1 - \frac{r_-}{r} \right)^{\frac{2q^2}{3+q^2}} \quad (2.15)$$

$$e^{-2\Phi} = \text{const} \quad e^{-2\Sigma} = \left( 1 - \frac{r_-}{r} \right)^{\frac{6q}{3+q^2}}$$

These solutions have not been considered previously, but essentially coincide with the generalized GHS solutions[1-2], where now  $\Sigma$  plays the role of the dilaton. They describe asymptotically flat black holes with mass  $\frac{1}{2} \left( r_+ + \frac{3-q^2}{3+q^2} r_- \right)$  and charge  $Q_2^2 = \frac{3}{3+q^2} r_+ r_-$ .

A horizon occurs at  $r = r_+$  and a singularity at  $r = r_-$ . The extremal limit  $r_+ = r_-$  is achieved when  $a = b = c = k = 0$ .

Also the limit  $A = 1$  is special and describes non-asymptotically flat black holes. For  $A = 1$  one has, in fact,

$$\begin{aligned} e^{2\nu} &= (r - r_+) r^{\frac{3-q^2}{3+q^2}} & e^{2\rho} &= r^{\frac{2q^2}{3+q^2}} \\ e^{-2\Phi} &= \text{const} & e^{-2\Sigma} &= r^{\frac{6q}{3+q^2}} \end{aligned} \quad (2.16)$$

Metrics of this form have been investigated in [6].

*C. The case  $\chi' = \eta'$ .*

The last case in which exact solutions can be obtained is given by the condition  $\eta = \chi + \text{const}$ , which corresponds to the solutions found in [5]. Setting  $e^{2\eta} = K^2 e^{2\chi}$ , the field equations become

$$\chi'' = (2Q_1^2 + K^2 Q_2^2) e^{2\chi} = \left( Q_1^2 + \frac{3+q^2}{3} K^2 Q_2^2 \right) e^{2\chi} \quad (2.17)$$

Hence,  $K^2 = \frac{3}{q^2} \frac{Q_1^2}{Q_2^2} = \frac{3}{\lambda^2 q^2}$ , and

$$Q_1 e^\chi = \sqrt{\frac{q^2}{3+2q^2}} \frac{2bAe^{b\xi}}{1-A^2 e^{2b\xi}} \quad (2.18)$$

where  $b$  and  $A$  are integration constants. The constraint (1.5) reduces to

$$a^2 - \frac{3+q^2}{3+2q^2} b^2 - \frac{q^2}{3+2q^2} c^2 = 0 \quad (2.19)$$

The solution possesses a horizon if

$$a - \frac{3+q^2}{3+2q^2} b + \frac{q^2}{3+2q^2} c = 0 \quad (2.20)$$

From (2.19) and (2.20), one obtains  $a = b = -c$ .

In terms of the coordinate  $r$  defined above, choosing  $A$  such that  $Q_1 = \sqrt{\frac{q^2}{3+2q^2}} \frac{2aA}{1-A^2}$ , the metric functions read

$$\begin{aligned} e^{2\nu} &= \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{3}{3+2q^2}} & e^{2\rho} &= r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2q^2}{3+2q^2}} \\ e^{-2\Phi} &= \left(1 - \frac{r_-}{r}\right)^{\frac{2q^2}{3+2q^2}} & e^{-2\Sigma} &= (3/q^2 \lambda^2)^{\frac{3}{q}} \left(1 - \frac{r_-}{r}\right)^{\frac{6q}{3+2q^2}} \end{aligned} \quad (2.21)$$



(Notice that we have exchanged the definition of  $r_+$  and  $r_-$ ). These solutions describe asymptotically flat black holes of mass  $\frac{1}{2} \left( r_+ + \frac{3}{3+2q^2} r_- \right)$  and charge  $Q_1^2 = \frac{q^2}{3+2q^2} r_+ r_-$  [5]. Also in this case the extremal black holes are obtained for vanishing  $a$ ,  $b$  and  $c$ .

In the special case  $A = 1$ , the solutions reduce to

$$\begin{aligned} e^{2\nu} &= (r - r_+) r^{\frac{3}{3+2q^2}} & e^{2\rho} &= r^{\frac{2q^2}{3+2q^2}} \\ e^{-2\Phi} &= r^{\frac{2q^2}{3+2q^2}} & e^{-2\Sigma} &= (3/q^2 \lambda^2)^{\frac{3}{q}} r^{\frac{6q}{3+q^2}} \end{aligned} \quad (2.22)$$

and describe non-asymptotically flat black holes .

### 3. The dynamical system.

The dynamical system (1.12) is similar to analogous systems studied in several contexts [8], but differs from these because the critical points at finite distance lie on a compact curve. It is easy to see, in fact, that all the critical points at finite distance are placed at the intersection between the plane  $Z = 0$  and the hyperboloid  $P = 0$ , with  $P$  defined in (1.13), which (except in some degenerate cases) is an ellipse.

In particular, the plane  $Z = 0$  corresponds to the limit  $Q_1 = 0$ . The system is invariant under  $Z \rightarrow -Z$ , but the  $Z < 0$  half-space is simply a copy of the positive  $Z$  half-space and has no physical significance. Hence, we shall not consider it in the following.

The hyperboloid  $P = 0$  contains the trajectories corresponding to the limit  $Q_2 = 0$ . If  $B > 0$  it is one-sheeted, while it is two-sheeted if  $B < 0$ . We shall consider only the former case. It is easy to see, in fact, that when  $B < 0$ , the hyperboloid does not intersect the plane  $Z = 0$  and therefore there are no critical points at finite distance. It follows that the solutions of the dynamical system are of oscillatory type, and do not lead to reasonable black hole geometries. Moreover, the physically relevant solutions are those in the exterior of the hyperboloid, which corresponds to  $|Q_2|e^x > 0$ , i.e. to the external region of the black hole. Finally, we notice that in the limit  $B = 0$ , the hyperboloid reduces to a cone and the only critical point at finite distance is the origin of the coordinates. This limit corresponds to extremal black hole solutions.

As noted above, when  $B > 0$ , the intersection of the hyperboloid  $P = 0$  with the plane  $Z = 0$  is given by an ellipse. More precisely, for every

$$|X_0| \leq \sqrt{\frac{9B}{(3+2q^2)(3+q^2)}}$$

there is a critical point at  $X = X_0$ ,  $Y = Y_0$ ,  $Z = 0$ , where  $Y_0$  is given in terms of  $X_0$  by the solution of the quadratic equation

$$(3+q^2)X_0^2 + 6Y_0^2 - 6X_0Y_0 - 3B = 0 \quad (3.1)$$

The characteristic equation for small perturbations,

$$\begin{aligned} X &= X_0 + x, & |x| &\ll 1 \\ Y &= Y_0 + y, & |y| &\ll 1 \\ Z &= Z_0 + z, & |z| &\ll 1 \end{aligned}$$

has eigenvalues 0,  $2X_0$  and  $Y_0$ . Hence, each point in the  $Z = 0$  plane satisfying (3.1) with  $X_0 > 0$ ,  $Y_0 > 0$ , repels a 2-dimensional bunch of solutions in the full 3-dimensional phase space, while solutions of (3.1) with  $X_0 < 0$ ,  $Y_0 < 0$  are attractors. The points with  $X_0 > 0$ ,  $Y_0 < 0$  or  $X_0 < 0$ ,  $Y_0 > 0$  act as saddle points. The presence of a vanishing eigenvalue is due of course to the fact that there is a continuous set of critical points lying on a curve. The critical points correspond to  $\xi \rightarrow -\infty$  for trajectories starting from the ellipse, and to the limit  $\xi \rightarrow \infty$  for trajectories ending at the ellipse.

The trajectories in the  $Z = 0$  plane, which correspond to the exact solutions discussed in sec. 2.A, are given by lines of equation  $Y = \frac{1}{2}(X - k)$ , with  $k$  a constant. The lines which do not intersect the ellipse of critical points, i.e. those with  $|k| > \sqrt{B}$ , correspond to oscillatory behaviour of  $X$  and  $Y$  and are not of interest to us. Notice that the extremal trajectories, for which  $|k| = \sqrt{B}$  are tangent to the ellipse at  $X = 0$ .

The projection of the trajectories corresponding to the exact solutions of section 2.B from the hyperboloid to the  $Z = 0$  plane presents a similar phase portrait. The trajectories satisfy in this case the equation  $Y = \frac{3+q^2}{3}(X - k')$  and only trajectories with  $|k'| < \sqrt{B/3}$  cross the ellipse. The projections of the extremal trajectories corresponding to  $|k'| = \sqrt{B/3}$  are now tangent to the ellipse at  $Y = 0$ .

For completeness, we notice that the solutions of sect. 2.c are given by the hyperbola of equation

$$(3 + q^2)(3 + 2q^2)Z^2 - 3(3 + q^2)X^2 = -9B, \quad (3.2)$$

lying in the plane  $X = Y$ .

We pass now to consider the behaviour of the metric functions  $e^\rho$ ,  $e^\nu$ , for  $\xi \rightarrow -\infty$ . In this limit,  $e^{2\chi} \sim e^{2X_0\xi}$ ,  $e^{2\eta} \sim e^{2Y_0\xi}$ , and hence,

$$\begin{aligned} e^\rho &\sim \exp \left[ \frac{1}{3 + 2q^2} ((3 + 2q^2)a + q^2c - q^2X_0 - 3Y_0)\xi \right] \\ e^{2\nu} &\sim \exp \left[ \frac{2}{3 + 2q^2} (-q^2c + q^2X_0 + 3Y_0)\xi \right] \end{aligned} \quad (3.3)$$

In general, the radius  $e^\rho \rightarrow 0$  as  $\xi \rightarrow -\infty$ , except in the special case

$$(3 + 2q^2)a + q^2c - q^2X_0 - 3Y_0 = 0 \quad (3.4)$$

This equation, combined with (3.1), gives the only real solution  $X_0 = Y_0 = a = -c$ . In this case  $e^\rho \rightarrow \text{const}$  as  $\xi \rightarrow -\infty$ . When these conditions are not satisfied, the metric function  $e^{2\nu}$  is singular near the critical points, giving rise to a singularity as  $e^\rho \rightarrow 0$ .

Also when the relation (3.4) is satisfied, the metric function  $e^{2\nu}$  behaves singularly near the critical points, but this can be shown to be simply a coordinate singularity by computing the curvature invariants, which tend to a constant value as  $\xi \rightarrow -\infty$  when (3.4) holds. Therefore, all the trajectories starting from the point  $X_0 = Y_0 = \sqrt{\frac{3}{3+q^2}}B$  correspond to solutions with regular horizon, provided  $c = -a$ .

To complete the analysis of the phase space we must also investigate the nature of the critical points on the surface at infinity. This can be done by defining new coordinates  $u$ ,

$y$ , and  $z$  such that infinity corresponds to  $u \rightarrow 0$ :

$$u = \frac{1}{X}, \quad y = \frac{Y}{X}, \quad z = \frac{Z}{X}$$

Then eqs. (1.12) take the form

$$\begin{aligned} \dot{u} &= -(z^2 + 2p)u \\ \dot{y} &= -(z^2 + 2p)y + \frac{3+q^2}{3}z^2 + p \\ \dot{z} &= (y - z^2 - 2p)z \end{aligned} \tag{3.5}$$

where we have defined  $p = P/X^2$  and a dot denotes  $u d/d\xi$ . The critical points with  $u = 0$  can be classified in three categories:

- 1) Two critical points, which we denote  $L_{1,2}$  placed at  $y = 1/2$ ,  $z = 0$ , i.e.

$$X = \pm\infty, \quad Y = \frac{X}{2}, \quad Z = 0$$

These are the endpoints of the trajectories lying in the  $Z = 0$  plane. The analysis of stability shows that the point with  $X > 0$  (resp.  $X < 0$ ) acts as an attractor (resp. repeller) both on the trajectories coming from finite distance and on the two-dimensional bunch of trajectories lying on the surface at infinity.

- 2) Two critical points  $M_{1,2}$  lie at  $y = (3 + q^2)/3$ ,  $z^2 = (3 + q^2)/3$ , i.e.

$$X = \pm\infty, \quad Y = \frac{3+q^2}{3}X, \quad Z = \sqrt{\frac{3+q^2}{3}}X$$

These are the endpoints of the trajectories lying on the hyperboloid  $P = 0$ . The analysis of stability shows that also in this case the point with  $X > 0$  (resp.  $X < 0$ ) attracts (resp. repels) both the trajectories coming from finite distance and those lying on the surface at infinity.

- 3) Two critical points  $N_{1,2}$  lie at  $y = 1$ ,  $z^2 = 3/(3 + 2q^2)$ , i.e.

$$X = \pm\infty, \quad Y = X, \quad Z = \sqrt{\frac{3}{3+2q^2}}X$$

These are the endpoints of the hyperbola (3.2) in the  $X = Y$  plane. The points with  $X > 0$  (resp.  $X < 0$ ) act as attractors (resp. repellers) on the trajectories coming from finite distance and as saddle points on the trajectories at infinity.

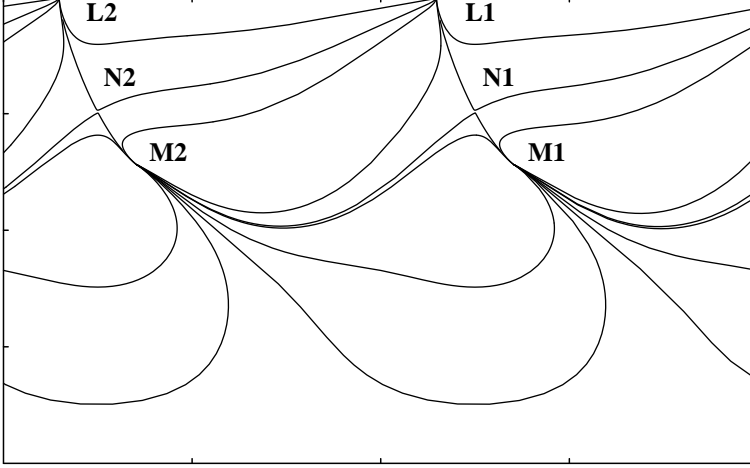


Fig. 1: The phase space at infinity.

In fig. 1 we sketch the pattern of trajectories on the surface at infinity. The point at infinity is reached for  $\xi \rightarrow \xi_0$ , where  $\xi_0$  is a finite constant. It is easy to see that for  $\xi \rightarrow \xi_0$ , the functions  $\chi$  and  $\eta$  behave as

$$e^\chi \sim |\xi - \xi_0|^{1/v_0} \quad e^\eta \sim |\xi - \xi_0|^{y_0/v_0}$$

where  $v_0 \equiv z_0^2 + 2p_0$ , the subscript 0 indicating the value taken at the critical points. Hence, if  $\xi_0 \neq 0$ , for  $\xi \rightarrow \xi_0$ , the metric functions behave as

$$e^\nu \sim e^{-\rho} \sim |\xi - \xi_0|^{\frac{3}{3+2q^2}(y_0 + \frac{q^2}{3})v_0^{-1}} \quad (3.6)$$

$$e^\Phi \sim |\xi - \xi_0|^{\frac{3}{3+2q^2}(y_0 - \frac{3+q^2}{3})v_0^{-1}} \quad e^\Sigma \sim |\xi - \xi_0|^{\frac{3q}{3+2q^2}(1-2y_0)v_0^{-1}}$$

The following picture of the phase space emerges: a family of trajectories start at the ellipse and end at one of the critical points  $L_1$ ,  $M_1$ ,  $N_1$ . Another family of trajectories start at one of the critical points  $L_2$ ,  $M_2$  or  $N_2$  and end at the ellipse. Moreover, there are trajectories which never intersect the ellipse, connecting the critical points at  $X = -\infty$  to those at  $X = +\infty$ . Of all the trajectories, only those starting at points of the ellipse such that  $X_0 = Y_0$  can correspond to regular solutions.

For completeness, we observe that most of the trajectories lying in the interior of the hyperboloid join  $M_1$  to  $M_2$ , but we shall not study them in detail because they are devoid of physical significance.

#### 4. Discussion.

We finally discuss the implications of the phase space portrait of the previous section on the physical properties of the solutions. For this purpose, it is useful to define a new radial coordinate  $r$  such that  $dr = e^{2\zeta} d\xi$ , as in sec. 2. One has:

$$r = \frac{r_+ - r_- e^{2a\xi}}{1 - e^{2a\xi}} \quad (4.1)$$

where we have defined  $r_+ = 2a(1 - e^{-2a\xi_0})^{-1}$ ,  $r_- = 2ae^{-2a\xi_0}(1 - e^{-2a\xi_0})^{-1}$ . In this way it is easy to identify the range of variation of  $\xi$  with the corresponding physical regions of the spacetime.

For  $\xi \rightarrow \mp\infty$ ,  $r \rightarrow r_{\pm}$ , while for  $\xi \rightarrow 0$ ,  $r \rightarrow \infty$ . Moreover, for  $\xi \rightarrow \xi_0$ , which without loss of generality we shall assume non-negative,  $r \rightarrow 0$ , except when  $\xi_0$  vanishes.

If  $\xi_0 \neq 0$ , we can identify the trajectories starting at the ellipse and ending at the point  $\xi_0$  with the exterior region of the black hole  $r > r_+$ . If the condition (3.4) is satisfied, these solutions possess a regular horizon. Moreover, they are asymptotically flat, since  $e^{2\chi}$  and  $e^{2\eta}$  tend to a constant as  $\xi \rightarrow 0$ . One can calculate the behaviour of these solutions as  $r \rightarrow r_+$ . From (3.3) and (3.4), one sees that for regular solutions  $e^{2\nu} \sim e^{2a\xi}$  and hence,  $e^{2\nu} \sim (r - r_+)$  for  $r \rightarrow r_+$ . In the same way one can see that the scalar fields are constant in that limit. It may be noticed that eq. (3.6) implies that in the unphysical limit  $r \rightarrow 0$ ,  $e^{2\nu} \sim e^{-2\rho}$ .

With our conventions, the trajectories starting at  $X = -\infty$  and ending at the ellipse correspond to the unphysical region  $0 < r < r_-$ . Unfortunately, since  $r_-$  is in general a singularity, one cannot single out the trajectories corresponding to physical solutions by requiring the regularity of the curvature invariants near that point, as for  $r_+$ . Moreover, with our methods, we are not able to connect the solutions in the region  $r > r_+$  with those in  $r < r_+$  and then to discuss their behaviour at  $r = r_-$ . This may however be achieved by using numerical methods.

The case  $\xi_0 = 0$  needs a separate discussion. The solutions are no longer asymptotically flat, but their behaviour for  $r \rightarrow \infty$  can be obtained from the  $\xi \rightarrow 0$  limit, which in our case turns out to be

$$\begin{aligned} e^\nu &\sim |\xi|^{\frac{3}{3+2q^2}(y_0 + \frac{q^2}{3})v_0^{-1}} & e^\rho &\sim |\xi|^{1 - \frac{3}{3+2q^2}(y_0 + \frac{q^2}{3})v_0^{-1}} \\ e^\Phi &\sim |\xi|^{\frac{3}{3+2q^2}(y_0 - \frac{3+q^2}{3})v_0^{-1}} & e^\Sigma &\sim |\xi|^{\frac{3q}{3+2q^2}(1-2y_0)v_0^{-1}} \end{aligned}$$

Moreover, since for  $\xi \rightarrow 0$ ,  $r \sim |\xi|^{-1}$ , it follows that for  $r \rightarrow \infty$  the solutions behave in one of the following three ways, depending on the critical points where they terminate,

	$e^{2\nu}$	$e^{2\rho}$	$e^{-2\Phi}$	$e^{-2\Sigma}$
$L_{1,2}$	$r$	$r$	$r$	const
$M_{1,2}$	$r^{6/(3+q^2)}$	$r^{2q^2/(3+q^2)}$	const	$r^{6q/(3+q^2)}$
$N_{1,2}$	$r^{(6+2q^2)/(3+2q^2)}$	$r^{2q^2/(3+2q^2)}$	$r^{2q^2/(3+2q^2)}$	$r^{6q/(3+2q^2)}$

These patterns coincide with those of the exact solutions (2.10), (2.16) or (2.22): hence all solutions of the system (1.2) are either asymptotically flat or possess the same asymptotic behaviour as one of the exact non-flat solutions. Moreover, from the discussion of sect. 3 of the phase space at infinity, it follows that the points  $N_{1,2}$  are unstable, so that most trajectories of this class actually behave like the solutions (2.10) or (2.16) for  $r \rightarrow \infty$ .

As noticed above, the other relevant limit case,  $B = 0$ , corresponds to the extremal black hole limit. In this case, the only critical point at finite distance is the origin of coordinates, and all the eigenvalues of the linearized equations vanish. This degeneration corresponds to a power-law behaviour of the variables  $X$ ,  $Y$  and  $Z$  near the critical point:  $X \sim -\alpha\xi^{-1}$ ,  $y \sim -\beta\xi^{-1}$ ,  $Z \sim \xi^{-\beta}$ ,  $\sqrt{P} \sim \xi^{-\alpha}$ . One can easily see from the field equations that the only possible values for  $\alpha$  and  $\beta$  are  $\alpha = 1, \beta = \frac{1}{2}$ ,  $\alpha = 1, \beta = 1$ , and  $\alpha = \frac{3}{3+q^2}, \beta = 1$ , which coincide with those of the exact extremal solutions of section 2. One can also check numerically that only the values  $\alpha = 1, \beta = 1$  are stable, so that all the trajectories, except the exact ones, behave near the critical point at the origin like the solutions (2.21) (case C). This is interesting, because from the previous discussion we know that this limit corresponds to the horizon of the extremal black hole. Now, it is well known that in the cases A and C of section 2, the extremal "string" metric  $d\hat{s}^2 = e^{2\phi}ds^2$  has a "near-horizon" limit in which the metric function  $e^{2\rho}$  becomes constant [9], and hence the spacetime decouples in the direct product of two 2-dimensional spaces, while this is not true for case B. But since all solutions except A and B, behave like C near the horizon, we can conclude that solution B is the only one for which  $e^{2\rho}$  is not constant near the horizon.

Before concluding this section, it is important to remark that the qualitative properties of the phase space and hence of the solutions are unaffected by the value of the parameter  $q$ , which is therefore essentially irrelevant for our discussion.

## 5. Conclusions.

From the previous discussion results that there is a large class of asymptotically flat regular black hole solutions of the field equations (1.2). These are characterized by three parameters: mass, electric charge (or equivalently  $r_+$  and  $r_-$ , or  $a$  and  $\xi_0$ ), and a third parameter which classifies the different trajectories starting from the critical points  $X_0 = Y_0 = \sqrt{\frac{3}{3+q^2}}B$ ,  $Z_0 = 0$ . We conjecture that the third parameter can be related to (a combination of) the scalar charges of the dilaton and the modulus. This conjecture cannot be checked explicitly because only in a few special cases the solution can be written in an analytic form.

The presence of an independent scalar charge would represent a novelty in the context of the no-hair results. In fact, in the known cases of dilaton gravity with non-minimal dilaton-Maxwell coupling, even if the dilaton is non-trivial, its charge is not an independent parameter, but is related to the mass and electric charge of the black hole (secondary hair). In our case of two non-minimally coupled scalar fields, it seems instead that a new independent charge is needed in order to classify the solutions.

Another interesting result is that in the extremal limit all the solutions except the unphysical case of a minimally coupled dilaton, have the same behaviour near the horizon, decoupling into the product of two 2-dimensional spaces. This is interesting since such a behaviour is required in recent attempts of calculating black hole entropy by counting

microstates of a conformal field theory [10].

Finally, we have clarified the role of non-asymptotically flat solutions, which were first discussed in [6] in the case of ordinary dilaton gravity, and shown that in our model they form a two-parameter family whose asymptotic behaviour can assume only three possible forms.

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